ON ORDERED DIVISIBLE GROUPS(1)

BY

NORMAN L. ALLING(2)

Introduction and remarks. In the theory of η_α-sets three main theorems stand out: that an η_α-set is universal for totally ordered sets of power not exceeding $\aleph_\alpha$, that any two η_α-sets of power $\aleph_\alpha$ are isomorphic and that η_α-sets of power $\aleph_\alpha$ exist provided $\aleph_\alpha$ is a regular cardinal number and $\sum_{\delta<\alpha} 2^{\aleph_\delta} \leqq \aleph_\alpha$. L. Gillman and M. Jerison [6] have shown that a real closed (totally ordered) field which is an η_α-set is universal for totally ordered fields of power not exceeding $\aleph_\alpha$, provided $\alpha > 0$. Further, P. Erdös, L. Gillman and M. Henriksen [3] have shown that any two real closed fields which are η_α-sets of power $\aleph_\alpha$ are isomorphic, provided $\alpha > 0$. They also showed that real closed fields which are η_α-sets of power $\aleph_\alpha$ exist, assuming the continuum hypothesis, if $\alpha = 1$. The question of existence for $\alpha > 1$, even assuming $\aleph_\alpha$ to be regular and $\sum_{\delta<\alpha} 2^{\aleph_\delta} \leqq \aleph_\alpha$, appears to be open.

In this paper it will be shown that given $\alpha > 0$ an η_α-group (i.e., a totally ordered Abelian divisible group which is an η_α-set) is universal for totally ordered Abelian groups of power not exceeding $\aleph_\alpha$, that any two η_α-groups of power $\aleph_\alpha$ are isomorphic and finally that, given an η_α-set of power $\aleph_\alpha$, η_α-groups of power $\aleph_\alpha$ exist.

Background. Let α be an ordinal number. By $W(\alpha)$ is meant the set of all ordinal numbers δ such that $\delta < \alpha$. Let $\aleph_\alpha$ be a cardinal number. By ω_α is meant the least ordinal number such that $W(\omega_\alpha)$ is of power $\aleph_\alpha$. A cardinal number $\aleph_\alpha$ is called regular if given an ordinal $\pi < \omega_\alpha$ and given a family of sets $(S_\delta)_{\delta<\pi}$, each of power less than $\aleph_\alpha$, the power of $\bigcup_{\delta<\pi} S_\delta$ is less than $\aleph_\alpha$.

Let T and T' be totally ordered sets and let f be a mapping of T into T'. We will call f order preserving (order reversing) if $x, y \in T$ and $x \leqq y$ implies $f(x) \leqq f(y)$, $(f(x) \geqq f(y))$ and strictly order preserving (strictly order reversing) if f is order preserving (order reversing) and one-to-one. Let H and K be subsets of T. We will write $H < K$ if $h < k$ for every $h \in H$ and $k \in K$. Note, according to this definition, $\emptyset < K$ and $K < \emptyset$ for all subsets K of T including $\emptyset$. T is said to be an *η_α-set* if, given subsets H and K of T of power less than $\aleph_\alpha$ such that $H < K$ then there exists $t \in T$ such that $H < \{t\} < K$. Let T be an η_α-set and let H be a subset of T of power less than $\aleph_\alpha$. By definition, $\emptyset < H$ and $H < \emptyset$. Thus there exist $u, v \in T$ such that $\emptyset < \{u\} < H$ and

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$H<\{v\}<\emptyset$: i.e., $\{u\}<H<\{v\}$. Hence, the definition of an η_α-set given above is equivalent to the classical definition. Let T be an η_α-set, let H and K be subsets of T of power less than $\aleph_\alpha$ and let $L=\{t\in T: H<\{t\}<K\}$. It is clear that $|L|$, the cardinal number of L, is not less than $\aleph_\alpha$. The initial results on η_α-sets are to be found in [7]. For more recent results see [5] and [9].

Throughout this paper N will denote the set of positive integers, Z the integers and Q the rational numbers.

1. Let G be a group which is also a partially ordered set. G will be called a *right* (*left*) *partially ordered group* if given a, b and $c\in G$, $a<b$ implies $ac<bc$ ($ca<cb$). A group which is a right or a left partially ordered group may be referred to as a one-sided partially ordered group. A *partially ordered group* is a group which is both a right and a left partially ordered group. If G is a totally ordered set (i.e., a partially ordered set in which every pair of elements is order comparable), then "totally" may be substituted for "partially" in the definitions above.

Let T be a totally ordered set. Let $\Pi(T)$ denote the permutation group of T and let $A(T)$ denote the automorphism group of T (i.e., all strictly order preserving mappings of T onto T). The set $\Pi(T)$ may be partially ordered by defining $f<g$, $f, g\in\Pi(T)$, if $f(t)<g(t)$ for all $t\in T$. Under this order, the only order we will put on $\Pi(T)$, $\Pi(T)$ is a right partially ordered group. To show this let f, g and $h\in\Pi(T)$, $f<g$. Clearly $fh(t)=f(h(t))<g(h(t))=gh(t)$ for all $t\in T$; thus $fh<gh$. $A(T)$ is a subgroup of $\Pi(T)$ and therefore is a right partially ordered group. Furthermore, $A(T)$ is a left partially ordered group. To see this, let f, g and $h\in A(T)$, $f<g$. Remembering that h is necessarily strictly order preserving we see that $hf(t)=h(f(t))<h(g(t))=hg(t)$; thus $hf<hg$, proving the assertion above. Hence $A(T)$ is a partially ordered group.

Let G be a subgroup of a one-sided partially ordered group B. Let $C_B(G)$ denote the centralizer of G in B (i.e., $\{b\in B: bg=gb$ for all $g\in G\}$). Let $K_B(G)=\{b\in B: b\in C_B(G)$ and b order comparable with each element of $G\}$.

PROPOSITION 1.1. *Let B be a one-sided partially ordered group and let G and H be Abelian subgroups of B such that $H\subset K_B(G)$. Then HG is a totally ordered Abelian subgroup of B.*

Proof. Clearly HG is Abelian; thus it is a partially ordered group. Let $t=hg$, $h\in H$ and $g\in G$. To show that HG is totally ordered it suffices to show that t is order comparable with 1. Since $h\in K_B(G)$, h is order comparable with g^{-1}; hence hg is order comparable with 1, proving the proposition.

Let B be a group and let $S\subset B$. Let $n\in N$ and let $S^n=\{s^n: s\in S\}$. S will be called *divisible* if $S^n=S$ for all $n\in N$.

PROPOSITION 1.2. *Let B be a one-sided partially ordered group and let G be a totally ordered Abelian divisible subgroup of B. If $b\in K_B(G)$ then $b^Z\subset K_B(G)$.*

Proof. Let $n \in N$. Clearly $b^n \in C_B(G)$. Let $g \in G$ and let $h = g^{1/n}$. Since b is order comparable with h, b^n is order comparable with $h^n = g$; thus $b^n \in K_B(G)$. Clearly b^{-n} is also in $K_B(G)$. Hence $b^Z \subset K_B(G)$.

COROLLARY 1.1. *Let B be a one-sided partially ordered group and let G be a totally ordered Abelian divisible subgroup of B. If $b \in K_B(G)$ then Gb^Z is a totally ordered Abelian subgroup of B.*

That the condition of divisibility in Corollary 1.1 is crucial may be seen by considering the following example. Let

$$G = \{g_z : z \in Z \text{ and } g_z(x) = x + z, x \in R\}.$$

Clearly G is a totally ordered Abelian subgroup of $\Pi(R)$. It is also clear that G is not divisible. Let $b(x) = (1/8) \sin (2\pi x) + x + 1/2$. Clearly $b \in \Pi(R)$; in fact $b \in A(R)$. Furthermore $b \in K_{\Pi(R)}(G)$. However, $g_1(1/4) < b^2(1/4)$ and $b^2(3/4) < g_1(3/4)$. Thus b^2 is not order comparable with g_1. Hence Gb^Z is not totally ordered.

PROPOSITION 1.3. *Let T be a totally ordered set and let H be a left totally ordered subgroup of $\Pi(T)$ which is transitive over T. Then $H \subset A(T)$.*

Proof. Let $x, y \in T$, $x < y$. Since H is transitive over T, $y = u(x)$ for some $u \in H$; thus $x < u(x)$. Since H is totally ordered, $1 < u$. Let $h \in H$. Since H is a left totally ordered group, $h1 < hu$; thus $h(x) < hu(x) = h(y)$. Hence $h \in A(T)$, proving the proposition.

COROLLARY 1.2. *Let T be a totally ordered set and let G be a totally ordered Abelian divisible subgroup of $\Pi(T)$. Let $b \in K_{\Pi(T)}(G)$ and assume that Gb^Z is transitive over T. Then Gb^Z is a totally ordered Abelian subgroup of $A(T)$.*

Proof. By Corollary 1.1, Gb^Z is a totally ordered Abelian subgroup of $\Pi(T)$. Since Gb^Z is transitive over T we may apply Proposition 1.3 and conclude that it is contained in $A(T)$.

PROPOSITION 1.4. *Let $(f_n)_{n \in N}$ be a sequence in a group G. Let $\phi(n) = $ l.c.m. $(1, \cdots, n)$ for each $n \in N$. Assume that $f_n^{\phi(n)/\phi(n-1)} = f_{n-1}$ for each $n \in N$ and let $H = \bigcup_{n \in N} f_n^Z$. H is Abelian and if it is torsion-free then it is divisible.*

Proof. Since $f_{n-1} \in f_n^Z$, $f_{n-1}^Z \subset f_n^Z$. Thus, H is the union of an ascending sequence of Abelian groups and is therefore Abelian. Clearly $f_1^{\phi(1)} = f_1$. Assume that $f_{n-1}^{\phi(n-1)} = f_1$. Clearly $f_n^{\phi(n)} = f_{n-1}^{\phi(n-1)} = f_1$. Let $k \in N$. We know that $\phi(k) = kt$ for some $t \in N$; thus $(f_k^t)^k = f_1$. Hence f_1 is divisible in H. Let $m, n \in N$. Thus there exists $x \in H$ such that $x^{m\phi(n)} = f_1$. We know that $f_1 = f_n^{\phi(n)}$; thus $x^{m\phi(n)} = f_n^{\phi(n)}$. Assume that H is torsion-free. Then $x^m = f_n$ and f_n is divisible by m; thus H is divisible, proving the proposition.

Let α be an ordinal number. A one-sided partially ordered group B will be called *α-hyper-divisible* if given a totally ordered Abelian divisible subgroup G of B of power less than $\aleph_\alpha$ then $K_B(G)$ is divisible.

PROPOSITION 1.5. *Let G be a totally ordered Abelian divisible subgroup of a one-sided partially ordered group B. Then $(K_B(G))^n \subset K_B(G)$ for all $n \in N$.*

Proof. Let $n \in N$ and let $f \in (K_B(G))^n$; thus $f = b^n$, $b \in K_B(G)$. By Proposition 1.2, $b^n \in K_B(G)$, proving the proposition.

COROLLARY 1.3. *For a one-sided partially ordered group B to be α-hyper-divisible it is necessary and sufficient that given a totally ordered Abelian divisible subgroup G of B, of power less than $\aleph_\alpha$, $n \in N$ and $f \in K_B(G)$ there exists $t \in K_B(G)$ such that $f = t^n$.*

PROPOSITION 1.6. *Let G be a totally ordered Abelian divisible subgroup of power less than $\aleph_\alpha$, of an α-hyper-divisible one-sided partially ordered group B and let $f \in K_B(G)$. There exists a totally ordered Abelian divisible subgroup G' of B such that $G \subset G'$, $f \in G'$ and $|G'| \leqq \aleph_0 |G|$.*

Proof. Let $\phi(n) = \text{l.c.m.}\ (1, \cdots, n)$, $n \in N$. Let $f_1 = f$ and assume f_{n-1} has been chosen in $K_B(G)$. Since B is α-hyper-divisible there exists $f_n \in K_B(G)$ such that $f_n^{\phi(n)/\phi(n-1)} = f_{n-1}$. Let $H = \bigcup_{n \in N} f_n^Z$.

Clearly H is Abelian. Since $f_n \in K_B(G)$, f_n is order comparable with 1; thus H is torsion-free. Hence, by Proposition 1.4, H is divisible. By Proposition 1.2, $f_n^Z \subset K_B(G)$; thus $H \subset K_B(G)$. Let $G' = HG$. By Proposition 1.1, G' is totally ordered and Abelian. Since both H and G are divisible, G' is divisible. Clearly $G \subset G'$. Since $f = f_1$, $f \in G'$. $|G'| \leqq |H| \times |G| = \aleph_0 |G|$, proving the proposition.

Let H be an (additive) Abelian group and let U be a subgroup of H. If H is torsion-free then U is pure if and only if $u = ny$, $u \in U$, $n \in N$ and $y \in H$ implies $y \in U$. The following two propositions will be of use in §2.

PROPOSITION 1.7. *Let U be a subgroup of a torsion-free (additive) Abelian group H. There exists a subset U' of H such that:*

1. $|U'| \leqq |U| \aleph_0$.
2. *U' is the smallest pure subgroup of H containing U.*
3. *If f is an isomorphism of U into an (additive) Abelian divisible group G then f extends to a unique isomorphism F of U' into G.*
4. *If H and G are totally ordered, if G is Abelian and divisible and if f is a strictly order preserving isomorphism of U into G, then f can be extended to a strictly order preserving isomorphism F of U' into G. Further, F is unique.*

Proof. Let $U' = \{h \in H : nh \in U \text{ for some } n \in N\}$. Clearly $|U'| \leqq |U| \aleph_0$ and $U \subset U'$. Let $x, y \in U'$. Then there exists $n, m \in N$ such that $nx, my \in U$. Clearly $nm(x+y) = m(nx) + n(my) \in U$; thus $x + y \in U'$. Clearly $n(-x) \in U$; thus $-x \in U'$ and U' is a group. Let V be a pure subgroup of H which contains U. We know that $nx \in U$; thus $nx \in V$. Since V is pure and H is torsion-free, $x \in V$; thus $U' \subset V$, proving 2. Proceeding to 3, let $F(x) = (1/n)f(nx)$. Since f is an isomorphism F is well defined. Clearly $F(x+y) = (1/nm)f(nm(x+y)) = (1/nm)f(nmx) + (1/nm)f(nmy) = F(x) + F(y)$. That F is unique may be seen by the following: let F' be an isomorphism of U' into G which extends f. Clearly $nF'(x) = F'(nx) = f(nx)$; thus $F'(x) = (1/n)f(nx) = F(x)$.

Given the conditions stated in 4, let F be the extension of f to U'. To show that F, an isomorphism, is a strictly order preserving mapping it suffices to show that $x > 0$ implies $F(x) > 0$. Since $x > 0$, $nx > 0$. Since f is strictly order preserving, $f(nx) > 0$. $F(x) = (1/n)f(nx) > 0$, proving the proposition.

The group U' will be referred to as the *purification of U in H*(3).

2. Let P and P' be partially ordered sets. A mapping h of P into P' will be called *l.u.b.-continuous* if given a nonempty totally ordered set $T \subset P$ such that l.u.b. T exists, then l.u.b. $h(T)$ exists and equals h (l.u.b. T). Clearly if h is l.u.b.-continuous then h is order preserving.

Let α be an ordinal number, let M be a set and let h be a mapping of a partially ordered set P into 2^M, the set of all subsets of M. Let $P_\alpha = \{p \in P: |h(p)| < \aleph_\alpha\}$. The pair (P, h) will be called *α-extendible* if given $p \in P_\alpha$ and $x \in M$ then there exists $p' \in P_\alpha$ such that $p \leqq p'$ and $x \in h(p')$.

A partially ordered set in which every nonempty totally ordered subset has a least upper bound will be called *inductive*.

LEMMA 2.1. *Let P be an inductive partially ordered set and let h be a l.u.b.-continuous mapping of P into 2^M, where M is a set of power $\aleph_\delta$. Let $\delta \leqq \alpha$ and assume that $\aleph_\alpha$ is a regular cardinal. If $P_\alpha \neq \emptyset$ and if (P, h) is α-extendible, then there exists $f \in P$ such that $h(f) = M$.*

Proof. Since we are concerned exclusively with the cardinality of M we may let $M = W(\omega_\delta)$.

By an *admissible family* we shall mean a family $(f_\epsilon)_{\epsilon<\pi}$ of elements of P_α such that:

1. π is an ordinal number, $\pi \leqq \omega_\delta$.
2. The mapping $\epsilon \to f_\epsilon$ is order preserving.
3. $W(\epsilon) \subset h(f_\epsilon)$ for all $\epsilon < \pi$.

By the *length* of an admissible family $(f_\epsilon)_{\epsilon<\pi}$ will be meant the ordinal π.

PROPOSITION 2.1. *If the length of an admissible family is π and if $\pi < \omega_\delta$ then the family can be extended to an admissible family of length $\pi + 1$.*

Proof. Let $(f_\epsilon)_{\epsilon<\pi}$ be an admissible family and assume that $\pi < \omega_\delta$.

Assume first that $\pi = 0$. Since $P_\alpha \neq \emptyset$ there exists $f_0 \in P_\alpha$. Clearly $W(0) = \emptyset \subset h(f_0)$.

(3) The referee has observed that U' can be characterized as being the subgroup U' of H containing U such that U'/U is the torsion subgroup of H/U.

Assume now that $\pi > 0$ and that π is a limit ordinal. Let $f_\pi = \text{l.u.b.}\ (f_\epsilon)_{\epsilon<\pi}$. Since P is inductive f_π is a well defined element of P. Since h is l.u.b.-continuous, $h(f_\pi) = \bigcup_{\epsilon<\pi} h(f_\epsilon)$. Thus $|h(f_\pi)| \leqq \sum_{\epsilon<\pi} |h(f_\epsilon)|$. For each $\epsilon<\pi$, $f_\epsilon \in P_\alpha$-thus $|h(f_\epsilon)| < \aleph_\alpha$. By assumption $\pi < \omega_\delta$ and $\omega_\delta \leqq \omega_\alpha$; thus $\pi < \omega_\alpha$. Since $\aleph_\alpha$ is regular, $\sum_{\epsilon<\pi} |h(f_\epsilon)| < \aleph_\alpha$; thus $f_\pi \in P_\alpha$. Since π is a limit ordinal, $W(\pi) = \bigcup_{\epsilon<\pi} W(\epsilon)$. Further, $\bigcup_{\epsilon<\pi} W(\epsilon) \subset \bigcup_{\epsilon<\pi} h(f_\epsilon) = h(f_\pi)$; thus $W(\pi) \subset h(f_\pi)$.

Assume finally that $\pi > 0$ and that π is a nonlimit ordinal: i.e., $\pi = \epsilon+1$, for some ordinal ϵ; then $f_\epsilon \in P_\alpha$. Since (P, h) is α-extendible, there exists $f_\pi \in P_\alpha$ such that $f_\epsilon \leqq f_\pi$ and $\epsilon \in h(f_\pi)$. Since h is l.u.b.-continuous, h is order preserving; thus $h(f_\epsilon) \subset h(f_\pi)$. Thus $W(\pi) = W(\epsilon) \cup \{\epsilon\} \subset h(f_\pi)$. Hence $W(\pi) \subset h(f_\pi)$.

Thus in each case $(f_\epsilon)_{\epsilon<\pi}$ can be extended to an admissible family of length $\pi+1$, proving the proposition.

An admissible family of length π may be thought of as a mapping from $W(\pi)$ into P_α. Let Γ be the set of all admissible families. Let a, b be in Γ. We will write $a \subset b$ if b is an extension of a. Under inclusion Γ is partially ordered. Let T be a nonempty totally ordered subset of Γ and let $t = \bigcup_{a\in T} a$. Clearly $t \in \Gamma$; thus Γ is inductive. By Zorn's Lemma there exists a maximal element $m \in \Gamma$. Were the length of m less than ω_δ then, by Proposition 2.1, we could extend m to a larger admissible family, which is absurd. Hence m is of length ω_δ. Let $f = \bigcup_{\epsilon<\omega_\delta} m(\epsilon)$. Since P is inductive, f is in P. Since ω_δ is a limit ordinal, since h is l.u.b.-continuous and since condition 3 holds for m, $M = W(\omega_\delta) = \bigcup_{\epsilon<\omega_\delta} W(\epsilon) \subset \bigcup_{\epsilon<\omega_\delta} hm(\epsilon) = h(f)$. But $h(f) \subset M$; thus $h(f) = M$, proving the lemma.

A stronger version of Lemma 2.1 can be proved in which the condition that $\delta \leqq \alpha$ and that $\aleph_\alpha$ is regular is replaced by the assumption that $\delta \leqq cf(\alpha)$. (For a definition of $cf(\alpha)$ see, e.g., [10].)

Corollary 2.1. *Let M be a set of power $\aleph_\delta$. Assume that $\aleph_\alpha$ is regular and that $\delta \leqq \alpha$. Let h be an l.u.b.-continuous mapping of an inductive partially ordered set P into 2^M. If (P, h) is α-extendible then given $f_0 \in P_\alpha$ there exists $f \in P$ such that $f_0 \leqq f$ and $h(f) = M$.*

Proof. Let $P' = \{p \in P; f_0 \leqq p\}$. Clearly $P'_\alpha \neq \emptyset$. Further P' is an inductive partially ordered set. Restricting h to P' we see that it is l.u.b.-continuous. Clearly (P', h) is α-extendible. Thus, by Lemma 2.1, there exists $f \in P'$ such that $h(f) = M$. Since $f \in P'$, $f_0 \leqq f$.

Let A and B be sets and let $A_0 \subset A$. A mapping f_0 of A_0 into B will be called a *partially defined mapping* of A into B. Let P be a set of partially defined mappings of A into B. Let Dp denote the domain of p and let Rp denote the range of p. Let $p, p' \in P$. If p' is an extension of p we will write $p \subset p'$. Under this order P is a partially ordered set. Further the mappings $p \rightarrow Dp$, $p \rightarrow Rp$ and $p \rightarrow (D \times R)p = (Dp) \times (Rp)$ of P into 2^A, 2^B and $2^A \times 2^B$, respectively, are order preserving. Let T be a nonempty totally ordered subset of P. Let $t \in T$ and let $g(t)$ denote the graph of t. Under the order on P, given $p, p' \in P$, $p \subset p'$ if and only if $g(p) \subset g(p')$. Clearly $\bigcup_{t \in T} g(t)$ is the graph of a partially defined mapping of u of A into B; u will be called the *union of* T. The set P will be called *union-inductive* if given a nonempty totally ordered subset T of P then the union of T is in P. The following is easily proved.

PROPOSITION 2.2. *Let P be a set of partially defined mappings of A into B, partially ordered by extension. If P is union-inductive then P is inductive and D, R and $D \times R$ are l.u.b.-continuous.*

THEOREM 2.1. *Let T and G be totally ordered (additive) Abelian groups, G being furthermore divisible and an η_α-set. Let P be the set of all partially defined strictly order preserving isomorphisms p of T into G for which Dp is a pure subgroup of T. Then P is union-inductive and (P, D) is α-extendible, provided $\alpha > 0$.*

Proof. Since T and G are totally ordered they are torsion-free. In particular 0 is a pure subgroup of T; thus $P_\alpha \neq \emptyset$. It is well known (see, e.g., [8]) that the union of a nonempty totally ordered set of pure subgroups is itself pure; thus P is union-inductive.

Let $\alpha > 0$. It remains to show that (P, D) is α-extendible. Let $p \in P_\alpha$ and let $x \in T$. If $x \in Dp$ let $p' = p$. Assume that $x \notin Dp$. Let

$$H_n = (1/n)p(]-\infty, nx[\cap Dp), \qquad n \in N,$$
$$K_n = (1/n)p(]nx, +\infty[\cap Dp), \qquad n \in N,$$

and let $H = \bigcup_{n \in N} H_n$ and let $K = \bigcup_{n \in N} K_n$.

We observe that $H < K$. To show this let $u \in H$ and let $u' \in K$. Clearly $u \in H_n$ and $u' \in K_{n'}$ for some $n, n' \in N$; thus $u = (1/n)p(t)$ and $u' = (1/n')p(t')$ for some t and t' in Dp such that $t < nx$ and $n'x < t'$. Clearly $n't < nn'x < nt'$. Thus $n'p(t) < np(t')$ and $u < u'$; hence we have proved that $H < K$.

Since $p \in P_\alpha$, $|Dp| < \aleph_\alpha$. Since $\alpha > 0, |H| + |K| \leqq \aleph_0 |Dp| < \aleph_0 \aleph_\alpha = \aleph_\alpha$. Since G is an η_α-set there exists $y \in G$ such that $H < \{y\} < K$. Let $U = Dp + Zx$. Since Dp is pure the sum is direct, thus there exists a homomorphism p' of U into G which extends p and such that $p'(x) = y$.

Let $u \in U$ such that $u > 0$. Since Dp is pure u is uniquely expressible as $t + zx$, $t \in Dp$ and $z \in Z$. If $z = 0$, $p'(u) = p(t) > 0$, since p is strictly order preserving. Since $u > 0$, we know that $zx > -t$ and that $t > -zx$. If $z > 0$, using the first of these two inequalities, we see that $(1/z)p(-t) \in H_z \subset H < \{y\}$; thus $p(t) + zy > 0$. If $z < 0$, using the second inequality, we see that $(1/-z)p(t) \in K_{-z} \subset K > \{y\}$; thus $p(t) + zy > 0$. Hence $p'(u) > 0$, proving that p' is a strictly order preserving isomorphism of U into G.

The group U, which is Dp', may not be a pure subgroup of T. Let V be the purification of U in T. By Proposition 1.7, p' extends to s, a strictly order preserving isomorphism of V into G. Since V is pure in T, $s \in P$. Since $\alpha > 0$ and $p \in P_\alpha$, $|V| \leqq |U|\aleph_0 = (|Dp| + \aleph_0)\aleph_0 < \aleph_\alpha$; thus $s \in P_\alpha$. Clearly $x \in U \subset V$ and $p \subset s$. Hence (P, D) is α-extendible, proving the theorem.

THEOREM A. *Let T be a totally ordered (additive) Abelian group of power not exceeding $\aleph_\alpha$ and let G be a totally ordered (additive) Abelian divisible group which is an η_α-set. Let f_0 be a strictly order preserving isomorphism of a subgroup U, of power less than $\aleph_\alpha$, of T into G. If $\alpha > 0$ then f_0 extends to a strictly order preserving isomorphism of T into G.*

Proof. Assume $\alpha > 0$. From the existence of G we may assume that $\aleph_\alpha$ is regular(4). Let T_0 be the purification of U in T and let p_0 be the extension of f_0 to T_0. We know that $|T_0| \leqq \aleph_0 |U| < \aleph_\alpha$, since $\alpha > 0$. Let P be the set of all partially defined strictly order preserving isomorphisms p of T into G for which Dp is a pure subgroup of T. Clearly $p_0 \in P_\alpha$. By Theorem 2.1, P is union-inductive and (P, D) is α-extendible. By Proposition 2.2, P is inductive and D is l.u.b.-continuous. By Corollary 2.1, there exists $f \in P$ such that $p_0 \subset f$ and $Df = T$: that is, f is an order preserving isomorphism of T into G which extends f_0, and the theorem is proved.

Theorem A is not true if we allow α to be 0. To see this, consider the following example. Let $T = \{a + b2^{1/2} : a, b \in Q\}$. Clearly T is a countable set and yet it can not be mapped isomorphically into Q, an η_0-set.

Let A and B be sets and let P be a set of partially defined mappings of A into B.

PROPOSITION 2.3. *If P is a set of partially defined one-to-one mappings of A into B such that (P, D) and (P^{-1}, D) are α-extendible then $(P, D \times R)$ is α-extendible.*

Proof. In the abstract case P_α is determined by P, α and h. Since the elements of P are one-to-one mappings, the sets P_α as defined by D, R and $D \times R$ are identical.

Let $p \in P_\alpha$ and let $(x, y) \in A \times B$. Since (P, D) is α-extendible, there exists $t \in P_\alpha$ such that $p \subset t$ and $x \in Dt$. Clearly $t^{-1} \in (P^{-1})_\alpha$. By hypothesis (P^{-1}, D) is α-extendible. Thus there exists $w \in (P^{-1})_\alpha$ such that $t^{-1} \subset w$ and such that $y \in Dw$. Since $(P^{-1})_\alpha = (P_\alpha)^{-1}$, $w^{-1} = s \in P_\alpha$. Thus $p \subset s$ and $(x, y) \in (D \times R)s$; hence the proposition is proved.

We can now prove the second of our major results.

THEOREM B. *Let G and G' be totally ordered (additive) Abelian divisible groups which are η_α-sets of power $\aleph_\alpha$. Any order preserving isomorphism f_0 of a subgroup U of G of power less than $\aleph_\alpha$ into G' extends to an order preserving isomorphism of G onto G', provided $\alpha > 0$.*

(4) Hausdorff [7] has shown that if $\aleph_\alpha$ is singular then G is an $\eta_{\alpha+1}$-set. $\aleph_{\alpha+1}$ is necessarily regular.

Proof. Assume $\alpha>0$ and let P be the set of all partially defined order preserving isomorphisms p of G into G' for which Dp is a pure subgroup of G. Clearly $P_\alpha\neq\emptyset$. By Theorem 2.1, (P, D) is α-extendible. Let $p\in P$. Since Dp is a pure subgroup of G (a divisible group), Dp is divisible. Since p is an isomorphism of Dp onto Rp, Rp is divisible and is therefore pure. By Theorem 2.1, (P^{-1}, D) is α-extendible. By Proposition 2.3, $(P, D\times R)$ is α-extendible. Theorem 2.1 tells us that P is union-inductive. By Proposition 2.2, P is inductive and $D\times R$ is l.u.b.-continuous.

Let V be the purification of U in G and let p_0 be the extension of f_0 to V, which exists and is unique by Proposition 1.7. Clearly $p_0\in P$. By Proposition 1.7, $|V|\leqq|U|\aleph_0$. By hypothesis $|U|<\aleph_\alpha$ and $\alpha>0$; thus $p_0\in P_\alpha$.

Since G is an η_α-set of power $\aleph_\alpha$, $\aleph_\alpha$ is a regular cardinal. By Corollary 2.1, there exists $f\in P$ such that $p_0\subset f$ and such that $(D\times R)f=G\times G'$: i.e., f extends f_0 and is an order preserving isomorphism of G onto G', proving the theorem.

The remainder of this paper will be devoted to a proof of the existence of totally ordered (additive) Abelian divisible groups which are η_α-sets of power $\aleph_\alpha$.

Let E be an η_α-set of power $\aleph_\alpha$. As remarked earlier, $\aleph_\alpha$ is necessarily regular.

THEOREM 2.2. *Let G be a totally ordered Abelian subgroup of $A(E)$ of power less than $\aleph_\alpha$ and let $H, K\subset G$ such that $H\cup K=G$ and $H<K$. Let $\{H, K\}$ be the set of all partially defined strictly order preserving mappings p of E into E such that the following conditions hold:*

(2.1) If $t\in Dp$ and $g\in G$ then $g(t)\in Dp$ and $gp(t)=pg(t)$. (Hence $GDp=Dp$.)

(2.2) If $t\in Dp$ then $H(t)<p(t)<K(t)$.

Then $(\{H, K\}, D)$ is α-extendible.

Proof. Let $p\in\{H, K\}_\alpha$ and let $e\in E$. If $e\in Dp$ let $p'=p$. Assume that $e\notin Dp$. Since $H<K$, $H(e)<K(e)$. Let $U=]-\infty, e[\cap Dp$ and let $V=]e, +\infty[\cap Dp$. Since p is strictly order preserving, $p(U)<p(V)$. Let $g\in H$ and let $t\in V$. Since $e<t$, $g(e)<g(t)$. By condition (2.2), $g(t)<p(t)$; thus $g(e)<p(t)$ and therefore $H(e)<p(V)$. Similarly, $p(U)<K(e)$. Hence $H(e)\cup p(U)<K(e)\cup p(V)$. The sets on either side of this inequality are of power less than $\aleph_\alpha$. Since E is an η_α-set, there exists a point $s(e)$ in E such that $H(e)\cup p(U)<\{s(e)\}<K(e)\cup p(V)$. By condition (2.1), since $e\notin Dp$, $G(e)\cap Dp=\emptyset$. Let s extend p and be defined on $G(e)$ as follows: $s(g(e))=g(s(e))$ for all $g\in G$.

We wish to show that s is strictly order preserving. To show this let $x, x'\in Ds$, $x<x'$. Assume first that $x=g(e)$ and $x'=g'(e)$, $g, g'\in G$. Since G is totally ordered, g and g' are uniquely determined by x and x'. Since $x<x'$, $g<g'$; thus $s(x)=s(g(e))=gs(e)<g's(e)=s(g'(e))=s(x')$. Now let $x=g(e)$, $g\in G$, and $x'\in Dp$. Since $x<x'$, $e<g^{-1}(x')$. By condition (2.1), $g^{-1}(x')\in Dp$. Furthermore $g^{-1}(x')\in V$. We know that $s(e)<p(V)$; thus $s(e)<pg^{-1}(x')$. By

condition (2.1), $pg^{-1}(x')=g^{-1}p(x')$. Thus $s(x)=sg(e)=gs(e)<p(x')=s(x')$. If $x\in Dp$ and $x'=g(e)$, $g\in G$, then $g^{-1}(x)<e$. Thus $g^{-1}(x)\in U$ and $pg^{-1}(x)<s(e)$. Hence $s(x)<s(x')$. Since p is strictly order preserving and since s is an extension of p, x, $x'\in Dp$ implies $s(x)<s(x')$. Hence s is strictly order preserving.

Clearly s satisfies condition (2.1). As for condition (2.2), let $h\in H$ and $k\in K$ and let $t\in Ds$. We know that p satisfies condition (2.2); thus $t\in Dp$ implies $H(t)<\{s(t)\}<K(t)$. Let $t=g(e)$, $g\in G$. We know that $H(e)<\{s(e)\}<K(e)$; thus $h(e)<s(e)<k(e)$. Clearly $h(t)=hg(e)=gh(e)<gs(e)=s(t)<gk(e)=kg(e)=k(t)$. Hence s satisfies condition (2.2), and therefore $s\in\{H, K\}$.

We know that $Ds=Dp\cup G(e)$; thus $|Ds|\leqq|Dp|+|G|$. Since $p\in P_\alpha$, $|Dp|<\aleph_\alpha$. By hypothesis $|G|<\aleph_\alpha$; thus $|Ds|<\aleph_\alpha$ and $s\in\{H, K\}_\alpha$. Clearly $e\in Ds$, thus the theorem is proved.

Continuing in the notation of Theorem 2.2, we have the following:

PROPOSITION 2.4. *The set $\{H, K\}$ of partially defined mappings of E into E is union-inductive.*

Proof. Let T be a nonempty totally ordered subset of $\{H, K\}$ and let s be the union of T. Let $x\in Ds$ and let $g\in G$. Clearly there exists $p\in T$ such that $x\in Dp$. Thus $g(x)\in Dp\subset Ds$ and $gs(x)=sg(x)$. Further $H(x)<s(x)<K(x)$; thus $s\in\{H, K\}$, proving the proposition.

PROPOSITION 2.5. $\{H, K\}^{-1}=\{K^{-1}, H^{-1}\}$.

Proof. The mappings in $\{H, K\}$ are strictly order preserving, therefore they are one-to-one mappings; thus $\{H, K\}^{-1}$ is well defined. Since G is a totally ordered group $K^{-1}\cup H^{-1}=G$ and $K^{-1}<H^{-1}$; thus $\{K^{-1}, H^{-1}\}$ is well defined and is, indeed, the set of all partially defined strictly order preserving mappings s of E into E such that (2.1) holds and such that the following holds:

(2.3) If $w\in Ds$ then $K^{-1}(w)<s(w)<H^{-1}(w)$.

STATEMENT 2.1. Let p be a partially defined one-to-one mapping of E into E which satisfies condition (2.1). Then p^{-1} satisfies condition (2.1).

Proof. Let $t\in Dp$, $g\in G$ and let $w=p(t)$. Clearly $w\in Rp=Dp^{-1}$. By condition (2.1), $g(t)\in Dp$ and $gp(t)=pg(t)$. Thus $g(w)=gp(t)=pg(t)\in Rp=Dp^{-1}$. Further, $g(w)=g(pp^{-1}(w))=pgp^{-1}(w)$; thus $p^{-1}g(w)=gp^{-1}(w)$. Hence p^{-1} satisfies condition (2.1), proving the statement.

STATEMENT 2.2. Let p be a partially defined one-to-one mapping of E into E. p satisfies condition (2.2) if and only if p^{-1} satisfies condition (2.3).

Proof. p satisfies condition (2.2), if and only if $H(t)<\{p(t)\}<K(t)$ for all $t\in Dp$: i.e., $K^{-1}(p(t))<\{t\}<H^{-1}(p(t))$. Let $w=p(t)$. We have shown that p satisfies condition (2.2) if and only if p^{-1} satisfies condition (2.3), proving the statement.

Returning now to the thesis of Proposition 2.5, let $p\in\{H, K\}$. Clearly $p^{-1}\in\{H, K\}^{-1}$. According to Statement 2.1 and Statement 2.2, p^{-1} satisfies conditions (2.1) and (2.3); thus $p^{-1}\in\{K^{-1}, H^{-1}\}$. Now let $s\in\{K^{-1}, H^{-1}\}$. Thus s satisfies conditions (2.1) and (2.3). By Statement 2.2, s^{-1} satisfies condition (2.2); thus $s^{-1}\in\{H, K\}$ and hence $s\in\{H, K\}^{-1}$, proving the proposition.

COROLLARY 2.2. *The pair* $(\{H, K\}, D\times R)$ *is* α*-extendible.*

Proof. According to Theorem 2.2, $(\{H, K\}, D)$ and $(\{K^{-1}, H^{-1}\}, D)$ are α-extendible. By Proposition 2.5, $\{K^{-1}, H^{-1}\}=\{H, K\}^{-1}$. Applying Proposition 2.3 we see that $(\{H, K\}, D\times R)$ is α-extendible, proving the corollary.

We now can prove a lemma which will be used explicitly in the proof of the existence theorem.

LEMMA 2.2. *Let* E *be an* η_α*-set of power* $\aleph_\alpha$, $\alpha>0$, *and let* G *be a totally ordered Abelian subgroup of* $A(E)$ *of power less than* $\aleph_\alpha$. *Let* x *and* y *be in* E. *There exists* $f\in K_{A(E)}(G)$ *such that* $y=f(x)$.

Proof. If there exists $g\in G$ such that $y=g(x)$ then let $f=g$. Clearly $g\in K_{A(E)}(G)$. If no such g exists let $H=\{g\in G\colon g(x)<y\}$ and let $K=\{g\in G\colon g(x)>y\}$. Clearly $H\cup K=G$ and $H<K$. By Proposition 2.4, $\{H, K\}$ is union-inductive. By Proposition 2.2, $\{H, K\}$ is inductive and $D\times R$ is l.u.b.-continuous. By Corollary 2.2, $(\{H, K\}, D\times R)$ is α-extendible. Let $u(g(x))=g(y)$, for each $g\in G$. Since G is a totally ordered group, u is strictly order preserving. Let $t\in Du$. Since $Du=G(x)$, $t=g'(x)$ for some $g'\in G$. Let $g\in G$. Clearly $g(t)=gg'(x)\in G(x)=Du$ and $gu(t)=gug'(x)=gg'(y)=ugg'(x)=ug(t)$; thus u satisfies condition (2.1). To show that u satisfies condition (2.2), it suffices to show that $H(g'(x))<\{g'(y)\}<K(g'(x))$. Since G is Abelian it suffices to show that $H(x)<\{y\}<K(x)$, which is true; thus $u\in\{H, K\}$. By hypothesis $|G|<\aleph_\alpha$; thus $|Du|<\aleph_\alpha$ and $u\in\{H, K\}_\alpha$. By Corollary 2.1, there exists $f\in\{H, K\}$ such that $(D\times R)f=E\times E$ and such that $u\subset f$. Thus $f\in A(E)$ and, by conditions (2.1) and (2.2), $f\in K_{A(E)}(G)$. Further $y=u(x)=f(x)$, proving the lemma.

LEMMA 2.3. *Let* E *be an* η_α*-set of power* $\aleph_\alpha$ *and assume that* $\alpha>0$. *The group* $A(E)$ *is an* α*-hyper-divisible partially ordered group.*

A proof of this lemma will be given in §3.

THEOREM C. *Let* E *be an* η_α*-set of power* $\aleph_\alpha$; $\alpha>0$. *Given a totally ordered Abelian divisible subgroup* G_0 *of* $A(E)$, *of power less than* $\aleph_\alpha$, *it is contained in a totally ordered Abelian divisible subgroup* G *of* $A(E)$ *which is simply transitive over* E; *thus* G *is an* η_α*-set of power* $\aleph_\alpha$.

Proof. Let P be the set of all totally ordered Abelian divisible subgroups of $A(E)$. Let P be ordered by inclusion. Let x be a fixed element in E. Let $v(G)=G(x)$ for each $G\in P$. Thus v maps P into 2^E. Clearly v is union-inductive; thus P is inductive and v is l.u.b.-continuous.

Let $G\in P_\alpha$ and let $y\in E$. By Lemma 2.2, there exists $f\in K_{A(E)}(G)$ such that $y=f(x)$. By Lemma 2.3, $A(E)$ is α-hyper-divisible. According to Proposition 1.6, there exists a totally ordered Abelian divisible subgroup G' of $A(E)$ such that $G\subset G'$, $f\in G'$ and $|G'|\leq|G|\aleph_0$. Thus $G'\in P_\alpha$ and $y\in G'(x)=v(G')$. Hence (P, v) is α-extendible.

By hypothesis $G_0\in P_\alpha$. By Corollary 2.1, there exists $G\in P$ such that $v(G)=E$: i.e., $G(x)=E$. Let e, $e'\in E$. There exist g, $g'\in G$ such that $e=g(x)$ and $e'=g'(x)$; thus $g'g^{-1}(e)=e'$ proving that G is transitive over E. Since G is totally ordered, G is simply transitive over E. Finally, since G is totally ordered, G is order isomorphic with $G(x)=E$; thus G is a totally ordered Abelian divisible group which is an η_α-set of power $\aleph_\alpha$, proving the theorem.

Having assumed the existence of an η_α-set of power $\aleph_\alpha$ it has been shown that totally ordered Abelian divisible groups which are η_α-sets of power $\aleph_\alpha$ exist.

3. It remains to prove Lemma 2.3. Let E be an η_α-set of power $\aleph_\alpha$, $\alpha>0$. We must prove that $A(E)$ is α-hyper-divisible. By Corollary 1.3, it suffices to prove that given a totally ordered Abelian divisible subgroup G of $A(E)$, of power less than $\aleph_\alpha$, $n\in N$ and $f\in K_{A(E)}(G)$ then there exists $t\in K_{A(E)}(G)$ such that $f=t^n$.

If $n=1$ then there is nothing to prove. Assume that $n>1$. If $f\in G$ let $t=f^{1/n}$. Since G is divisible, such an element exists and is in G. Assume now that $f\notin G$. Since $f\in K_{A(E)}(G)$, f is order comparable with 1. Since $f\notin G$, $f\neq 1$; hence either f or f^{-1} is greater than 1. We may assume, without loss of generality, that $f>1$.

Let P be the set of all partially defined strictly order preserving mappings p of E into E such that the following conditions hold:

(3.1) p, f and the elements of G restricted to Dp are permutations of Dp.

(3.2) $p\in K_{\Pi(D_p)}(G|Dp)$(5).

(3.3) $f=p^n$ over Dp.

Clearly P is union-inductive. Thus P is inductive and D is l.u.b.-continuous. Our main task is to show that (P, D) is α-extendible. Having shown this, Lemma 2.1 will be invoked to prove the lemma.

Let $H=\{g\in G: g<f\}$ and let $K=\{g\in G: g>f\}$. Since $f\in K_{A(E)}(G)$ and $f\notin G$, $H\cup K=G$. Clearly $H<K$. Thus $H<\{f\}<K$. It is clear that (3.2) and (3.3) combine to give us:

(3.3′) $H^{1/n}<\{p\}<K^{1/n}$ over Dp.

(By $S^{1/n}$ is meant $\{s^{1/n}: s\in S\}$, where $S\subset G$.)

Let $p\in P_\alpha$ and let $u\in E$. If $u\in Dp$ let $p'=p$. Assume that $u\notin Dp$. An *admissible sequence* is a sequence $(e_j)_{0\leq j<m}$ of points in E such that:

(3.4) $u=e_0$.

(3.5) $H^{1/n}(e_{j-1})<\{e_j\}<K^{1/n}(e_{j-1})$, for $0<j<m$.

(3.6) $x<e_{j-1}<x'$, x, $x'\in Dp$ implies $p(x)<e_j<p(x')$, for $0<j<m$.

(5) $G|Dp=\{g|Dp: g\in G\}$, where by $g|Dp$ is meant the restriction of g to Dp.

(3.7) $K^{(j-n)/n}(f(e_0)) < \{e_j\} < H^{(j-n)/n}(f(e_0))$, for $0 \leqq j < m$.

(3.8) $e_j \neq gf^r(e_i)$, for every $g \in G$, $r \in Z$, for $0 \leqq i < j < m$.

Let $(e_j)_{0 \leqq j < m}$ be an admissible sequence. By the length of this sequence will be meant the number m. Clearly any initial segment of an admissible sequence is an admissible sequence.

PROPOSITION 3.1. *If $(e_j)_{0 \leqq j < m}$ is an admissible sequence, then $e_j \notin Dp$, $0 \leqq j < m$.*

Proof. By condition (3.4), $e_0 = u$ and $u \notin Dp$. Assume $e_{j-1} \notin Dp$, $0 < j < m$. Thus $\{x \in Dp : x < e_{j-1}\} \cup \{x \in Dp : x > e_{j-1}\} = Dp$. Since p is a permutation of Dp, $p\{x \in Dp : x < e_{j-1}\} \cup p\{x \in Dp : x > e_{j-1}\} = Dp$. By (3.6) e_j lies between these two sets whose union is Dp. Thus $e_j \notin Dp$, proving the proposition.

PROPOSITION 3.2. *Given an admissible sequence of length m, then $x < e_j < x'$, $x, x' \in Dp$, implies $p^{-j}(x) < e_0 < p^{-j}(x')$, $0 \leqq j < m$.*

Proof. Clearly the statement is true for $j = 0$. Let $0 < j < m$ and assume that the statement is true for $j-1$. Let $x \in Dp$ such that $x < e_j$ and assume for a moment that $p^{-1}(x) \geqq e_{j-1}$. By Proposition 3.1, $e_{j-1} \notin Dp$. By (3.1), $p^{-1}(x) \in Dp$; thus $p^{-1}(x) > e_{j-1}$. By (3.6), $pp^{-1}(x) > e_j$: i.e., $x > e_j$, which is absurd. Thus $p^{-1}(x) < e_{j-1}$. We have assumed that this implies $p^{-(j-1)}p^{-1}(x) < e_0$: i.e., $p^{-j}(x) < e_0$. By a similar argument it can be shown that $e_j < x'$, $x' \in Dp$, implies $e_0 < p^{-j}(x')$. Thus the proposition is proved.

LEMMA 3.1. *There exists an admissible sequence of length n.*

A proof of Lemma 3.1 will be given in §4.

Henceforth let $(e_j)_{0 \leqq j < n}$ be an admissible sequence of length n. Let m be an integer. Clearly m can be written as $rn + i$, $0 \leqq i < n$, r, $i \in Z$. Further, r and i are uniquely determined by m. If $r \neq 0$ let e_m be defined as $f^r(e_i)$; thus for all integers m, $e_m = f^r(e_i)$.

PROPOSITION 3.3. *Let $y = g(e_m)$ and $y' = g'(e_{m'})$, g, $g' \in G$ and m, $m' \in Z$. $y = y'$ if and only if $g = g'$ and $m = m'$.*

Proof. Assume that $y = y'$, $m = rn + i$, $m' = r'n + i'$ and $0 \leqq i$, $i' < n$. If $i \neq i'$ we may assume, without loss of generality, that $i > i'$. Clearly $e_i = g^{-1}g'f^{r'-r}(e_{i'})$, which is untenable by condition (3.8). Thus $i = i'$ and $f^{r-r'}(e_i) = g^{-1}g'(e_i)$. We know that $f \in K_{A(E)}(G)$. By Proposition 1.2, $f^Z \subset K_{A(E)}(G)$; thus $f^{r-r'} = g^{-1}g'$. Since $f \notin G$, $r - r' = 0$ and $g = g'$. If, on the other hand, $g = g'$ and $m = m'$ then $r = r'$ and $i = i'$; thus $y = y'$, proving the proposition.

Let $T = \{g(e_m) : g \in G \text{ and } m \in Z\}$. Since T is a subset of E, a totally ordered set, it is a totally ordered set. Clearly f and the elements of G act as permutations on T as they do on Dp. By Proposition 3.1, $(e_j)_{0 \leqq j < n} \cap Dp = \varnothing$;

thus $T\cap Dp=\varnothing$. Let $y\in T$. By Proposition 3.3, y can be written as $g(e_m)$ for one and only one pair (g, m), where $g\in G$ and $m\in Z$. Let p' extend p and let $p'(y)=g(e_{m+1})$ for each $y\in T$.

PROPOSITION 3.4. *p' is a partially defined mapping of E into E which satisfies conditions* (3.1)–(3.3).

Proof. Both Dp and T are subsets of E. Since p' acts as a permutation on both these sets, p' is a partially defined mapping of E into E. Further p', f and the elements of G, restricted to Dp', are permutations of Dp': i.e., p' satisfies condition (3.1).

Let g, $g'\in G$ and let $y=g(e_m)$, $m\in Z$. Clearly $p'g'(y)=p'g'g(e_m)=g'g(e_{m+1})=g'p'g(e_m)=g'p'(y)$; thus $p'g'=g'p'$ over T. Hence $p'\in C_{\Pi(Dp')}(G|Dp')$. Let $h\in H^{1/n}$ and let $k\in K^{1/n}$. Let $m=rn+i$, $0\leqq i<n$. If $i+1<n$ we may invoke condition (3.5) and conclude that $h(e_i)<e_{i+1}<k(e_i)$. If $i+1=n$ we may invoke condition (3.7) and conclude that $k^{-1}(f(e_0))<e_{n-1}<h^{-1}(f(e_0))$; thus $h(e_{n-1})<f(e_0)<k(e_{n-1})$. Remembering that $f(e_0)=e_n$ and that, in this case, $i=n-1$, we see that $h(e_i)<e_{i+1}<k(e_i)$. Thus this holds for all i, $0\leqq i<n$. Since $f\in K_{A(E)}(G)$ we may apply Proposition 1.2 and conclude that $f^r\in K_{A(E)}(G)$. Thus $h(f^r(e_i))<f^r(e_{i+1})<k(f^r(e_i))$. Since G is Abelian and totally ordered, $h(y)<p'(y)<k(y)$; hence $H^{1/n}<\{p'\}<K^{1/n}$ over T. By (3.3′), this also holds over Dp; thus it holds over Dp'. We know that $H\cup K=G$. Since G is divisible, $H^{1/n}\cup K^{1/n}=G$. We conclude that p' is order comparable with the elements of G: i.e., $p'\in K_{\Pi(Dp')}(G|Dp')$ and p' satisfies condition (3.2). Clearly $(p')^n(y)=g(e_{m+n})=gf(e_m)=fg(e_m)=f(y)$; thus p' satisfies condition (3.3), proving the proposition.

PROPOSITION 3.5. *The group $G'=G((p')^Z)$ is transitive over T.*

Proof. It has been shown that $p'\in K_{\Pi(Dp')}(G|Dp')$. Let $x\in T$. Clearly x may be written as $g(e_m)$, $g\in G$ and $m\in Z$. Let $g(p')^m=a$. Clearly $a\in G'$ and $x=a(e_0)$. Let $y\in T$. By an argument similar to the preceding, there exists $b\in G'$ such that $y=b(e_0)$. Thus $y=ba^{-1}(x)$ and $ba^{-1}\in G'$, proving the proposition.

COROLLARY 3.1. *p' restricted to T is strictly order preserving.*

Proof. By Proposition 3.5 and Corollary 1.2, $G'|T\subset A(T)$; thus in particular, $p'|T$ is in $A(T)$. Hence p' is strictly order preserving over T, proving the corollary.

PROPOSITION 3.6. *Let x, $y\in Dp$ and $t\in T$. If $x<t<y$ then $p'(x)<p'(t)<p'(y)$.*

Proof. Let $t=g(e_m)$, $g\in G$ and $m\in Z$. Let $m=rn+i$, $0\leqq i<n$; thus $t=f^rg(e_i)$. Let $v=g^{-1}f^{-r}(x)$ and $w=g^{-1}f^{-r}(y)$. Since $g^{-1}f^{-r}$ is strictly order preserving,

$v < e_i < w$. If $i = n-1$ we may apply Proposition 3.2 and conclude that $p^{-i}(v) < e_0 < p^{-i}(w)$. Thus $p'(v) = p^{i+1}p^{-i}(v) = fp^{-i}(v) < f(e_0) < fp^{-i}(w) = p'(w)$. Since $i = n-1$, $p'(e_i) = e_n = f(e_0)$; thus $p'(v) < p'(e_i) < p'(w)$. If $i < n-1$ we may invoke condition (3.6) and conclude that $p'(x) < p'(t) < p'(y)$, proving the proposition.

COROLLARY 3.2. *p' is strictly order preserving.*

Proof. p' extends p, which is strictly order preserving; thus p' is strictly order preserving over Dp. By Corollary 3.1, p' is strictly order preserving over T. Combining these results with Proposition 3.6, we see that p' is strictly order preserving over $Dp \cup T = Dp'$, proving the corollary.

Hence $p' \in P$. Clearly $|Dp'| = |Dp| + |T|$. Since $p \in P_\alpha$, $|Dp| < \aleph_\alpha$. By hypothesis $|G| < \aleph_\alpha$ and $\alpha > 0$; thus $|T| = \aleph_0 |G| < \aleph_\alpha$. Hence $|Dp'| < \aleph_\alpha$ and $p' \in P_\alpha$. Furthermore p' extends p and $u = e_0 \in T \subset Dp'$; thus (P, D) is α-extendible. Clearly the empty mapping is in P; thus $P_\alpha \neq \emptyset$.

By Lemma 2.1, there exists $t \in P$ such that $Dt = E$. By condition (3.1), t is a permutation of Dt: i.e., t is a permutation of E. Since $t \in P$, t is strictly order preserving; thus, by condition (3.2), $t \in K_{A(E)}(G)$. By condition (3.3), $f = t^n$ over E; thus Lemma 2.3 is proved.

4. The proof of Lemma 3.1 will be given in this section.

Let $e_0 = u$ and consider the sequence $\{e_0\}$. Clearly this sequence satisfies condition (3.4). In this case conditions (3.5), (3.6) and (3.8) are vacuous. In this case condition (3.7) states that $K^{-1}(f(e_0)) < \{e_0\} < H^{-1}(f(e_0))$; this is true since $H < \{f\} < K$. Thus $\{e_0\}$ is an admissible sequence of length 1.

LEMMA 4.1. *Any admissible sequence of length m can be extended to an admissible sequence of length $m+1$, provided $1 \leqq m < n$.*

Granting Lemma 4.1 for a moment, we see that Lemma 3.1 is proved.

LEMMA 4.2. *Let $(e_j)_{0 \leq j < m}$ be an admissible sequence of length m, $1 \leqq m < n$. Then*

$$L_0(m) = H^{1/n}(e_{m-1}) \cup p\{x \in Dp: x < e_{m-1}\} \cup K^{(m-n)/n}(f(e_0))$$
$$< K^{1/n}(e_{m-1}) \cup p\{x \in Dp: x > e_{m-1}\} \cup H^{(m-n)/n}(f(e_0)) = L_1(m).$$

This lemma will be proved shortly. Clearly $|L_i(m)| \leqq |G| + |Dp| + |G| < \aleph_\alpha$ for $i = 0, 1$. Since E is an η_α-set, there exist at least $\aleph_\alpha$ points $x \in E$ such that $L_0(m) < \{x\} < L_1(m)$. The set $S(m) = \{gf^r(e_i): g \in G,\ r \in Z \text{ and } 0 \leqq i < m\}$ is of power $|G|\aleph_0$. Since $\alpha > 0$, $|S(m)| < \aleph_\alpha$. Thus there exists e_m in E such that $L_0(m) < \{e_m\} < L_1(m)$ and such that $e_m \notin S(m)$: i.e., $(e_j)_{0 \leq j < m+1}$ is an admissible sequence of length $m+1$, proving Lemma 4.1.

It remains to prove Lemma 4.2. $L_0(m)$ is the union of three sets, as is $L_1(m)$. It will be shown, in Propositions 4.1 through 4.9, that each of the three sets, considered in Lemma 4.2, whose union is $L_0(m)$ is less than each of the three sets whose union is $L_1(m)$. Having shown this, Lemma 4.2 will be proved.

PROPOSITION 4.1. $H^{1/n}(e_{m-1})<K^{1/n}(e_{m-1})$.

This is true since $H<K$ and $n>0$.

PROPOSITION 4.2. *Given* $h\in H^{1/n}$ *and* $x\in Dp$ *such that* $e_{m-1}<x$ *then* $h(e_{m-1})<p(x)$.

Proof. By (3.3′), $H^{1/n}<\{p\}$; thus $h(e_{m-1})<h(x)<p(x)$, proving the proposition.

By symmetry we also have:

PROPOSITION 4.3. *Given* $k\in K^{1/n}$ *and* $x\in Dp$ *such that* $e_{m-1}>x$, *then* $k(e_{m-1})>p(x)$.

PROPOSITION 4.4. $H^{1/n}(e_{m-1})<H^{(m-n)/n}(f(e_0))$.

Before proving this it will be useful to establish the following:

STATEMENT 4.1. If q, $s\in Q$ such that $qs>0$ then $H^qH^s=H^{q+s}$ and $K^qK^s=K^{q+s}$.

Proof. $H^qH^s=\{x^qy^s: x,\ y\in H\}$; thus $H^{q+s}\subset H^qH^s$. Let x, $y\in H$, $x\leqq y$. Since $qs>0$, q and s have the same sign; thus x^qy^s lies between x^{q+s} and y^{q+s}, both of which are in H^{q+s}. Since G is divisible, H^{q+s} is an interval of G; thus $x^qy^s\in H^{q+s}$. Hence $H^qH^s\subset H^{q+s}$. A similar argument can be used to prove the second assertion, proving the statement.

Proof of Proposition 4.4. Let $h\in H^{1/n}$ and let $h'\in H^{(m-n)/n}$. By Statement 4.1, $h^{-1}h'\in H^{(m-1-n)/n}$. By condition (3.7), $\{e_{m-1}\}<H^{(m-1-n)/n}(f(e_0))$: i.e., $e_{m-1}<h^{-1}h'f(e_0)$. Hence $h(e_{m-1})<h'f(e_0)$, proving the proposition.

By symmetry we also have:

PROPOSITION 4.5. $K^{(m-n)/n}(f(e_0))<K^{1/n}(e_{m-1})$.

PROPOSITION 4.6. *Given* x, $x'\in Dp$ *such that* $x<e_{m-1}<x'$, *then* $p(x)<p(x')$.

This is true since p is strictly order preserving.

PROPOSITION 4.7. *Given* $h\in H^{(m-n)/n}$ *and* $x\in Dp$ *such that* $x<e_{m-1}$, *then* $p(x)<hf(e_0)$.

Proof. Applying Proposition 3.2 we see that $p^{1-m}(x)<e_0$. By condition (3.3′), $H^{1/n}<\{p\}$. Further $m<n$; thus $\{p^{m-n}\}<H^{(m-n)/n}$. Hence $p^{m-n}=p^mf^{-1}$ implies $p^m<hf$. Clearly $p(x)=p^mp^{1-m}(x)<hfp^{1-m}(x)<hf(e_0)$, proving the proposition.

By symmetry we have:

PROPOSITION 4.8. *Given* $x\in Dp$ *and* $k\in K^{(m-n)/n}$ *such that* $x>e_{m-1}$, *then* $p(x)>kf(e_0)$.

Since $H<K$ and $m<n$ we have:

PROPOSITION 4.9. $K^{(m-n)/n}(f(e_0))<H^{(m-n)/n}(f(e_0))$.

Thus Lemma 4.2 is proved.

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PURDUE UNIVERSITY,
LAFAYETTE, INDIANA